JOURNAL OF COMPUTATIONAL PHYSICS 77, 513–523 (1988)

High Order Difference Methods for Linear Variable Coefficient Parabolic Equation*

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Received January 23, 1987; revised June 29, 1987

In this paper, we derive $O(k^2+h^4)$, two-level, three-point finite-difference methods for the solution of the general linear variable coefficient parabolic equation in one dimension $u_t = a(x)u_{xx} + b(x)u_x + c(x)u$, $0 \leqslant x \leqslant 1$, $t>0$ under suitable initial and boundary conditions. The stability of the new schemes is examined using a linear stability analysis. In particular, we derive unconditionally stable $O(k^2+h^4)$ methods for the solution of the convection-diffusion type equation in cylindrical and spherical coordinates, viz. $u_t = u_{xx} + (\alpha/x)u_x + cu$, $0<\alpha<1$ or $\alpha = 1, 2$. These methods are tested or two examples.

1. INTRODUCTION

The numerical solution of the general variable coefficient parabolic partial differential equation is of great interest in physics and applied mathematics. An $O(k+h^2)$ symmetrical semi-implicit scheme for the general heat conduction equation was formulated by Livne and Glasner [1]. A comparative study of some explicit and implicit finite-difference schemes was done by Roberts and Selim [2] in the one-dimensional case. High order operator compact implicit methods for parabolic equations were discussed in [3, 4]. Varah [5] had studied the stability restrictions on second-order, three-level finite-difference schemes for parabolic equations. Some monotone difference schemes for diffusion-convection problems were discussed by Stoyan [6]. In [7–9] some classes of extrapolation finite-difference methods for the numerical solution of a constant-coefficient, one-dimensional homogeneous parabolic equation were derived. These methods are of $O(h^2)$ in space direction while high order in time direction is obtained by extrapolation. Cash [10] has shown that these methods can be interpreted as implicit Runge–Kutta formulas and derived two additional methods to solve the quasi-linear equation $u_t = au_{xx} + f(u, u_x)$, a constant. All these schemes are derived such

* This research is supported by the National Science and Engineering Council of Canada, operating Grants A3054 and the University of Saskatchewan President's NSERC.

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0021-9991/88 $3.00

581/77/2-16

that the corresponding methods are L_0 stable. These methods are useful in solving parabolic equations having high frequency components occurring in the solution. It is well known that A_0 stable methods like Crank–Nicolson methods perform poorly on such problems.

The numerical solution of the one-dimensional cylindrical heat conduction equation

$$u_t = u_{rr} + u_r/r \tag{1a}$$

was dealt by Mitchell and Pearce [11] and Iyengar and Mittal [12]. Mitchell and Pearce used the transformation $r = 2x^{1/2}$ and the mesh $x = i^2h$. They have derived two explicit schemes and an implicit scheme. This implicit scheme has the same stability restriction as one of the two explicit schemes. Iyengar and Mittal have derived an $O(k^2 + h^4)$, two-level, three-point implicit-difference scheme with the stability limit on the mesh ratio parameter $\lambda < \frac{242}{9}$. The truncation error of this scheme contains lower order derivatives which may entail some loss of accuracy because of the singularity at $r = 0$. So far no unconditionally stable, two-level, three-point implicit scheme of $O(k^2 + h^4)$ is known.

In this paper, we derive $O(k^2 + h^4)$, two-level, three-point difference methods for the solution of the one-dimensional parabolic equation

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u, \qquad 0 \leqslant x \leqslant 1,\ a(x) \geqslant 0, \tag{1b}$$

under suitable initial and boundary conditions. The differential equation may have a singularity at one or both end points. A typical example is the convection-diffusion equation in the cylindrical and spherical coordinate systems

$$u_t = u_{xx} + (\alpha/x)u_x + cu, \qquad c = \text{constant} \leqslant 0. \tag{1c}$$

For Eq. (1b) we have derived a method whose truncation error does not contain lower order derivatives with respect to x explicitly. Criteria for unconditional stability of the methods for (1b) are derived and applied to some test equations. The stability of the difference method in the case of the convection-diffusion type equation in cylindrical and spherical coordinates is discussed using the general stability conditions. This method is unconditionally stable for $c \leqslant 0$ for all α considered here. It is observed that this method produces the best order as $h \to 0$, $k \to 0$ in the neighborhood of the singularity $x = 0$ compared to the methods whose truncation error contains lower order derivatives. For a fixed λ (mesh ratio parameter), these methods behave like fourth-order methods. These methods are applied to two examples to demonstrate the fourth-order behaviour for fixed λ. All these methods produce a tridiagonal system for solution on each time level. Therefore, no additional computational effort is required in the implementation of the present method in comparison to the lower order methods.

2. Difference Methods

Let us first consider a procedure to construct high order difference schemes for the equation

$$a(x)u_{xx} + b(x)u_x + c(x)u = f(x). \tag{2}$$

Consider the uniform mesh $x_j = x_0 + jh$, $j = 0(1)N$, where $x_0 = 0$ and $x_N = 1$, and write the Taylor series expansion of a, b, c, u, and f about a nodal point x_j, which is taken as origin in the local coordinates, as

$$\begin{aligned} &a(x) = \sum \alpha_j x^j, \qquad b(x) = \sum \beta_j x^j, \qquad c(x) = \sum \gamma_j x^j, \\ &u(x) = \sum A_j x^j, \qquad f(x) = \sum d_j x^j. \end{aligned} \tag{3}$$

In order to obtain a difference scheme of fourth order we assume that $A_j \equiv 0$ for $j \geqslant 5$ and, similarly, for the other coefficients in (3). Substituting (3) in (2) and comparing the successive powers of x, we get

$$\begin{aligned} d_i = \sum_{j=0}^{i} &[(i-j+2)(i-j+1)\, A_{i-j+2}\alpha_j \\ &+ (i-j+1)\, A_{i-j+1}\beta_j + A_{i-j}\gamma_j], \qquad i = 0, 1, 2, 3, 4. \end{aligned} \tag{4}$$

We now write a linear combination of d_i as

$$s_0\, d_0 h^2 + s_1(d_1 h^3 + d_3 h^5) + s_2(d_2 h^4 + d_4 h^6) = \sum_{j=0}^{4} Q_j A_j h^j. \tag{5}$$

But, we know that

$$(A_1 h + A_3 h^3)_j = \delta_{2x} u_j/2 + O(h^5) \tag{6a}$$

$$(A_2 h^2 + A_4 h^4)_j = \delta_x^2 u_j/2 + O(h^6), \tag{6b}$$

where $\delta_{2x} u_j = u_{j+1} - u_{j-1}$ and $\delta_x^2 u_j = u_{j+1} - 2u_j + u_{j-1}$. Similar expressions hold for d_j's. Hence, we equate the coefficients of $A_1 h$, $A_3 h^3$ and $A_2 h^2$, $A_4 h^4$ on the right-hand side of (5). Solving these equations, we get

$$s_0 = p_1 q_2 - p_2 q_1, \quad s_1 = \beta_0 h q_2 - 2\alpha_0 p_2, \quad s_2 = 2\alpha_0 p_1 - \beta_0 h q_1, \tag{7}$$

where

$$\begin{aligned} p_1 &= 6\alpha_0 + 2(3\alpha_2 + \beta_1)h^2 - (\beta_3 + \gamma_2)h^4, \\ q_1 &= h[2(5\alpha_1 + \beta_0) - (2\alpha_3 + 2\beta_2 + \gamma_1)h^2] \\ p_2 &= h[3(2\alpha_1 + \beta_0) + 2(3\alpha_3 + \beta_2)h^2 - (\beta_4 + \gamma_3)h^4] \\ q_2 &= 12\alpha_0 + 2(5\alpha_2 + \beta_1)h^2 - (2\alpha_4 + 2\beta_3 + \gamma_2)h^4. \end{aligned} \tag{8}$$

Using (6a) and (6b) and similar expressions for d_j in (5), we write the method for the solution of (2) as

$$h^2[2s_0 + s_1\,\delta_{2x} + s_2\,\delta_x^2]\,f_j = [2Q_0 + Q_1\,\delta_{2x} + Q_2\,\delta_x^2]\,u_j, \tag{9}$$

where

$$\begin{aligned} Q_0 &= h^2[\gamma_0 s_0 + (\gamma_1 h + \gamma_3 h^3)s_1 + (\gamma_2 h^2 + \gamma_4 h^4)s_2] \\ Q_1 &= [6\alpha_0 + (6\alpha_2 + 3\beta_1 + \gamma_0)h^2]s_1 + [3(2\alpha_1 + \beta_0)h + (6\alpha_3 + 3\beta_2 + \gamma_1)h^3]s_2 \\ Q_2 &= 4(3\alpha_1 + \beta_0)hs_1 + [12\alpha_0 + (12\alpha_2 + 4\beta_1 + \gamma_0)h^2]s_2. \end{aligned} \tag{10}$$

The truncation error of the method (9) is

$$\text{T.E.} = \tfrac{3}{5}\,\alpha_0^2 h^6(\alpha_0\,D_x^6 u + 3\beta_0\,D_x^5 u)_j + \cdots, \tag{11}$$

where $D_x^p = \partial^p/\partial x^p$. Note that the leading term of the truncation error does not contain lower order derivatives of u and is independent of $c(x)$ and its derivatives. When $a(x)$, $b(x)$, and $c(x)$ are smooth functions or polynomials, it may not be necessary to include $s_1\,d_3 h^5 + s_2\,d_4 h^6$ in (5). Even if these terms are excluded from (5) the method corresponding to (9) is still of fourth order with s_0, s_1, s_2, etc. being defined as

$$\begin{aligned} &s_0 = p_1 q_2 - p_2 q_1, \quad s_1 = \beta_0 h q_2 - 2\alpha_0 p_2, \quad s_2 = 2\alpha_0 p_1 - \beta_0 h q_1, \\ &p_1 = 6\alpha_0 - (\beta_1 + \gamma_0)h^2,\ p_2 = h[3(2\alpha_1 + \beta_0) - (\beta_2 + \gamma_1)h^2],\ q_1 = -2(\alpha_1 + \beta_0)h, \\ &q_2 = 12\alpha_0 - (2\alpha_2 + 2\beta_1 + \gamma_0)h^2, \quad Q_0 = h^2[s_0\gamma_0 + s_1\gamma_1 h + s_2\gamma_2 h^2], \\ &Q_1 = 6\alpha_0 s_1 + 3(2\alpha_1 + \beta_0)s_2 h, \quad Q_2 = 12\alpha_0 s_2. \end{aligned} \tag{12}$$

The leading term of the truncation error now contains lower order derivatives and is given as

$$\begin{aligned} \text{T.E.} = \alpha_0 h^6[&\{5\alpha_0\,D_x^4 + 10(\beta_0 - 2\alpha_1)\,D_x^3\}\,f \\ &- 2\alpha_0(\alpha_0\,D_x^6 u + 3\beta_0\,D_x^5)u]/5. \end{aligned} \tag{13}$$

This method ((9) with (12)) has certain advantages. It does not contain α_j, β_j, γ_j for $j > 2$. The derivatives α_1, α_2, β_1, β_2, γ_1, and γ_2 need not be obtained by actual differentiation but may be replaced by the difference approximations $h\alpha_1 \approx \delta_{2x} a_j/2$, $h^2\alpha_2 \approx \delta_x^2 a_j/2$, etc. of second order. The resulting method would still retain its fourth-order accuracy. However, when the differential equation has a singularity at one or both of the end points of the interval [0, 1], this formula does not produce accurate results of fourth order compared to the method (9) with (10) in the neighbourhood of the singularity.

Replacing f_j in (9) by $(\partial u/\partial t)_j$ and using the approximation $(\partial u/\partial t)_j =$

$[\nabla_t/k(1-\frac{1}{2}\nabla_t)]u_j$, where k is the mesh size in the t-direction, we get the difference method for the solution of (1) as

$$\begin{aligned}&[(2s_0-\lambda Q_0)+(s_1-0.5\lambda Q_1)\,\delta_{2x}+(s_2-0.5\lambda Q_2)\,\delta_x^2]\,u_j^{n+1}\\&\quad=[(2s_0+\lambda Q_0)+(s_1+0.5\lambda Q_1)\,\delta_{2x}+(s_2+0.5\lambda Q_2)\,\delta_x^2]\,u_j^n,\end{aligned}\qquad(14)$$

where $\lambda=k/h^2$. We shall call (14) with (10) as "Method 1" and (14) with (12) as "Method 2." The truncation error of Method 1 is

$$T_1=\alpha_0^2[-12k^3u_{ttt}+\tfrac{3}{5}kh^4(\alpha_0\,D_x^6u+3\beta_0\,D_x^5u)]_j+\cdots\qquad(15)$$

and the truncation error of Method 2 is

$$T_2=-12\alpha_0^2k^3u_{ttt}+kh^4\alpha_0T^*+\cdots,\qquad(16)$$

where

$$T^*=[\{5\alpha_0\,D_x^4+10(\beta_0-2\alpha_1)\,D_x^3\}\,u_t-2\alpha_0(\alpha_0\,D_x^6+3\beta_0\,D_x^5)u]/5.$$

3. Stability of the Difference Schemes

The above methods can be written in the form

$$(a_1\,\delta_x^2+b_1\,\delta_{2x}+c_1)u_j^{n+1}=(a_2\,\delta_x^2+b_2\,\delta_{2x}+c_2)u_j^n.\qquad(17)$$

Using the von Neumann method, we find that the amplification factor of (17) satisfies

$$|\xi|^2=\frac{[c_2+2a_2(x-1)]^2+4b_2^2(1-x^2)}{[c_1+2a_1(x-1)]^2+4b_1^2(1-x^2)}=\frac{N}{N+(D-N)},\qquad(18)$$

where $x=\cos(\beta h)$, N and D stand for the numerator and denominator, respectively. Note that $N>0$ and $D>0$. For stability, we now require

$$\begin{aligned}g(x)&=D-N=c_1^2-c_2^2+4(1-x)\\&\quad\times[(c_2a_2-c_1a_1)+(1-x)(a_1^2-a_2^2)+(1+x)(b_1^2-b_2^2)]\geqslant 0.\end{aligned}\qquad(19)$$

Note that if $c_1=c_2$, then $g(x)=4(1-x)\,h(x)$ and hence we require $h(x)\geqslant 0$ since $|x|\leqslant 1$. But $h(x)$ is linear. Therefore, it is sufficient to have $h(-1)\geqslant 0$ and $h(1)\geqslant 0$. This gives the conditions

$$(a_1-a_2)[2(a_1+a_2)-c_1]\geqslant 0\qquad\text{and}\qquad c_1(a_2-a_1)+2(b_1^2-b_2^2)\geqslant 0.\qquad(20)$$

If $c_1\neq c_2$ in (17) then we are required to satisfy (19). In this case, we require $g(-1)\geqslant 0$, $g(1)\geqslant 0$, and in some cases $g_s\geqslant 0$, where g_s stands for the stationary value of $g(x)$.

For Methods 1 and 2 we have

$$g(x)=8\lambda\{[s_0-s_2(1-x)][(1-x)Q_2-Q_0]-Q_1s_1(1-x)(1+x)\}\geqslant 0. \quad (21)$$

We find that the method (14) is unconditionally stable when the following conditions are satisfied:

1. $c(x)\equiv 0$;

$$(s_0-2s_2)Q_2\geqslant 0 \quad \text{and} \quad Q_2s_0-2Q_1s_1\geqslant 0. \quad (22)$$

2. $c(x)\neq 0$;

(a) If $Q_2s_2-Q_1s_1\geqslant 0$, then $Q_0s_0\leqslant 0$ and $(s_0-2s_2)(2Q_2-Q_0)\geqslant 0$. (23a)

(b)(i) If $0<2Q_1s_1+Q_2s_0+Q_0s_2-4Q_2s_2<4(Q_1s_1-Q_2s_2)$, then

$$[(2Q_1s_1-Q_2s_0-Q_0s_2)^2+4Q_0s_0(Q_1s_1-Q_2s_2)]\leqslant 0; \quad (23b)$$

(ii) otherwise,

$$Q_2s_2-Q_1s_1<0, \quad Q_0s_0\leqslant 0, \quad (s_0-2s_2)(2Q_2-Q_0)\geqslant 0. \quad (23c)$$

(c) A simple sufficient condition is

$$s_0\geqslant 0, \quad s_0-2s_2\geqslant 0, \quad Q_0\leqslant 0, \quad 2Q_2-Q_0\geqslant 0, \quad Q_1s_1\leqslant 0. \quad (23d)$$

We now apply Methods 1 and 2 to the following test equations and determine the stability restrictions, using (21)–(23).

Test Equation 1. $u_t=au_{xx}+cu$; a, c constants, $a>0$, $c\leqslant 0$. Method 1 and 2: unconditionally stable.

Test Equation 2. $u_t=au_{xx}+bu_x$; a, b, constants, $a\geqslant 0$. As $a\to 0$, the scalar test equation becomes the model hyperbolic test equation with no decay in the solution. For such equations, we require our formula to be conservative ($|\xi|=1$), since there is no attenuation of the Fourier coefficients.

Method 1. Unconditionally stable for all b satisfying $b^2h^2\leqslant 24a^2$. The formula is conservative for the model hyperbolic equation.

Method 2. Unconditionally stable for all b. The formula is also conservative for the model hyperbolic equation.

We have also applied Methods 1 and 2 to the test equation $u_t=au_{xx}+bu_x+cu$; a, b, c constants, $a>0$, $c\neq 0$. The stability conditions are clumsy because of the presence of the parameters b, c, a and are not reported here. If $b(x)=0$ and $c(x)=0$ in (1), then Method 2 is unconditionally stable when $\alpha_0\geqslant 0$ and either $4\alpha_0\geqslant\alpha_2h^2$ or $\alpha_1^2>\alpha_0\alpha_2$ is satisfied. The first condition is usually satisfied in most of the cases. In general, given $a(x)$, $b(x)$, and $c(x)$ we can test the conditions (22)–(23) to determine whether the method (14) is unconditionally stable. This procedure is adopted in the following section.

4. Convection-Diffusion Equation in Polar Coordinates

Consider the general convection-diffusion equation

$$u_t = u_{xx} + (\alpha/x)u_x + cu, \qquad c = \text{constant} < 0, \tag{24}$$

where $0 < \alpha < 1$, $\alpha = 1$ or 2. For $\alpha = 1$ and 2, (24) corresponds to the convection-diffusion problem in the cylindrical and spherical coordinates, respectively. The initial and boundary conditions are

$$u(x, 0) = f_1(x), \quad u_x(0, t) = 0, \quad u(R, t) = g_1(t). \tag{25}$$

Substituting $a(x) = 1$, $b(x) = \alpha/x$, and $c(x) = c$ in Method 1, we get

$$\begin{aligned} &s_0 = 3[12 - \alpha(6+\alpha)p^2 + \alpha(4+\alpha)p^4], \quad s_1 = \alpha p[3 - (2+\alpha)p^2 + (1+\alpha)p^4], \\ &s_2 = 6 - \alpha(2+\alpha)p^2 + \alpha(1+\alpha)p^4, \quad Q_1 = Q_1^* + ch^2 s_1, \quad Q_2 = Q_2^* + ch^2 s_2, \\ &Q_0 = ch^2 s_0, \quad Q_1^* = 3\alpha p[12 - (\alpha^2 + 7\alpha - 2)p^2 + (1+\alpha)(2+\alpha)p^4], \\ &Q_2^* = 12[6 - 4\alpha p^2 + \alpha(1+\alpha)p^4], \quad p = h/x_j. \end{aligned} \tag{26}$$

For $\alpha = 2$, we find $s_0 = 6s_2$, $s_1 = ps_2$, $Q_1 = (12 + ch^2)ps_2$, $Q_2 = (12 + ch^2)s_2$, and $Q_0 = 6ch^2 s_2$ so that the factor s_2 can be cancelled and the method has a simple form. Using the above stability conditions, we find that the method (14) with (26) is unconditionally stable for all $c \leqslant 0$ and for all α considered here. The truncation error in the Method 1 now becomes

$$\text{T.E.} = -6u_{ttt}k^3 + \frac{3}{10}kh^4\left(D_x^6 u + \frac{3\alpha}{x}D_x^5 u\right) + \cdots. \tag{27}$$

Consider now the case $c = 0$. As $x \to 0$, we get from (24)

$$u_t = (1+\alpha)u_{xx}. \tag{28}$$

For $\alpha = 1$, we use the $O(k^2 + h^4)$ scheme (see [12]),

$$(5 + 12\lambda)u_0^{n+1} + (1 - 12\lambda)u_1^{n+1} = (5 - 12\lambda)u_0^n + (1 + 12\lambda)u_1^n, \tag{29}$$

valid at $j = 0$, where we have used the condition $u_x(0, t) = 0$. Similarly, a $O(k^2 + h^4)$ scheme valid at $j = 0$, when $\alpha = 2$ is

$$(5 + 18\lambda)u_0^{n+1} + (1 - 18\lambda)u_1^{n+1} = (5 - 18\lambda)u_0^n + (1 + 18\lambda)u_1^n. \tag{30}$$

Application of these schemes requires the solution of an $N \times N$ tridiagonal system at each time level. Recently, in [13], a fourth-order difference method was derived for the solution of the singular two-point boundary value problem

$$u_{xx} + (\alpha/x)u_x = f(x, u). \tag{31}$$

This method can also be generalized, to obtain an $O(k^2 + h^4)$ unconditionally stable scheme, to solve Eq. (24) when $c = 0$. We may write this method as

$$\begin{aligned}[t] &[(s_3 - 6\lambda s_1)\,\delta_x^2 + (s_4 - 6\lambda s_2)\,\delta_{2x} + 12]\,u_j^{n+1} \\ &\quad = [(s_3 + 6\lambda s_1)\,\delta_x^2 + (s_4 + 6\lambda s_2)\,\delta_{2x} + 12]\,u_j^n, \end{aligned} \tag{32}$$

where

$$s_1 = 1 - s_5 p^2 + s_5 s_6 p^4, \quad s_2 = p(\alpha + s_5 p^2 - s_5 s_6 p^4)/2, \quad s_3 = 1 - s_5 p^2/5,$$
$$s_4 = p[\alpha - s_5(2-\alpha)\,p^2/5]/2, \quad s_5 = \alpha(2-\alpha)/12, \quad s_6 = (\alpha^2 - 24)/60.$$

The truncation error in the method (32) is

$$\text{T.E.} = -k^3 u_{ttt} + \frac{kh^4}{20}\left[D_x^6 u + \frac{3\alpha}{x} D_x^5 u - 2(3+\alpha)(2-\alpha)\,D_x^4 u + \frac{6(2-\alpha)}{x^2} u_{xxt}\right] + \cdots. \tag{33}$$

Note that the truncation errors (27) and (33) are same for $\alpha = 2$. However (for method (32)), when $\alpha = 2$, we find tht $s_3 - s_4 = 0$ and $s_1 - s_2 = 0$ for $j = 1$. This means that the point $j = 0$ does not enter the difference equations and the system degenerates to a $(N-1) \times (N-1)$ system. The solution at $j = 0$ can be obtained by interpolation using the numerical solutions at $j = 1, 2$, etc. As in [13] we may use

$$u_0^{n+1} = (144u_1^{n+1} - 108u_2^{n+1} + 48u_3^{n+1} - 9u_4^{n+1})/75. \tag{34}$$

5. Computational Experiments

Computations reveal that the solutions obtained by the methods (14) with (26), and (32) are almost identical in the following problems. Therefore, the results obtained by using (14) with (26) are reported.

Example 1. $\alpha = 1$, $c = 0$ in (24) with

$$u(x, 0) = J_0(\alpha_0 x),\ 0 \leqslant x \leqslant 1; \qquad u_x(0, t) = 0; \qquad u(1, t) = 0,$$

where α_0 is the first root of $J_0(\alpha_0) = 0$. The exact solution is $u(x, t) = J_0(\alpha_0 x)\exp(-\alpha_0^2 t)$. This problem is solved using (14) with (26), and (32) for $h = 0.1$ and 0.05. The integration is done up to $t = 3.0$ for $\lambda = 0.5$, 1, 3, 5. Small values of λ

are also used to test whether the methods are of fourth order as $h \to 0$, $k \to 0$ (for fixed λ). Even though a number of fourth-order methods (for a fixed λ) can be constructed using the above procedure, it is found that (14) with (26) produced the best order as $h \to 0$, $k \to 0$ in the neighbourhood of $x = 0$. Computational results reported in Table I, along with the results of [12], show the fourth-order convergence (for a fixed λ) of our methods.

EXAMPLE 2. $\alpha = 2$, $c = 0$ in (24) with

$$u(x, 0) = 1 - x^2, 0 \leqslant x \leqslant 1, \qquad u_x(0, t) = 0, \qquad u(1, t) = 0.$$

The exact solution is

$$u(x, t) = \frac{12}{\pi^3 x} \sum_1^{\infty} \frac{(-1)^{n-1}}{n^3} \sin(n\pi x) \exp(-n^2\pi^2 t).$$

This problem is solved using (14) with (26), and (32) for $h = 0.1$ and 0.05. Orders of the methods are again tested using small values of λ. The results are given in Table I. Computations show the fourth-order convergence (for fixed λ) of our methods.

Absolute errors at $r = 0$ for small t and small λ, for both Examples 1 and 2, are given in Table II along with the results obtained by the Crank–Nicolson $O(k^2 + h^2)$ scheme. These results also exhibit the $O(h^4)$ behaviour of Method 1 on $r = 0$ even for small λ and t.

TABLE I

Maximum Absolute Errors in the Solution

		Example 1		Ref. [12]	Example 2	
λ	t	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.1$	$h = 0.05$
0.5	0.75	0.259(−5)	0.160(−6)	—	0.905(−6)	0.573(−7)
	1.50	0.683(−7)	0.424(−8)	0.810(−7)	0.109(−8)	0.688(−10)
	3.00	0.234(−10)	0.146(−11)	0.279(−10)	0.807(−15)	0.508(−16)
1.0	0.75	0.143(−4)	0.898(−6)	—	0.423(−5)	0.265(−6)
	1.50	0.376(−6)	0.235(−7)	0.405(−6)	0.514(−8)	0.323(−9)
	3.00	0.129(−9)	0.805(−11)	0.139(−9)	0.380(−14)	0.240(−1)
3.0	0.75	0.140(−3)	0.879(−5)	—	0.393(−4)	0.249(−5)
	1.50	0.366(−5)	0.230(−6)	0.375(−5)	0.460(−7)	0.303(−8)
	3.00	0.124(−8)	0.785(−10)	0.127(−8)	0.978(−12)	0.225(−14)
5.0	0.75	0.391(−3)	0.246(−4)	—	0.106(−3)	0.693(−5)
	1.50	0.103(−4)	0.642(−6)	0.103(−4)	0.340(−6)	0.842(−8)
	3.00	0.682(−8)	0.219(−9)	0.347(−8)	0.840(−10)	0.620(−14)

TABLE II

Absolute Error at $r = 0$

		Example 1				Example 2			
		Method 1		Second-order Scheme		Method 1		Second-order Scheme	
λ	$t\backslash h$	0.1	0.05	0.1	0.05	0.1	0.05	0.1	0.05
0.1	0.05	0.194(−4)	0.132(−5)	0.115(−2)	0.294(−3)	0.151(−4)	0.962(−6)	0.753(−3)	0.190(−3)
	0.10	0.180(−4)	0.121(−5)	0.176(−2)	0.447(−3)	0.112(−5)	0.784(−7)	0.286(−2)	0.728(−3)
	0.15	0.159(−4)	0.106(−5)	0.200(−2)	0.505(−3)	0.952(−5)	0.599(−6)	0.318(−2)	0.801(−3)
0.2	0.05	0.176(−4)	0.121(−5)	0.115(−2)	0.294(−3)	0.418(−5)	0.277(−6)	0.763(−3)	0.191(−3)
	0.10	0.153(−4)	0.104(−5)	0.176(−2)	0.446(−3)	0.108(−5)	0.589(−7)	0.286(−2)	0.728(−3)
	0.15	0.129(−4)	0.871(−6)	0.200(−2)	0.505(−3)	0.142(−5)	0.931(−7)	0.317(−2)	0.800(−3)

It is found that the maximum error generally occurs on $r=0$ or h for all t, λ and the difference between the errors on $r=0$, h is marginal. From these numerical results, it can be seen that the proposed higher order methods produce accurate results and the error decreases at a faster rate than the lower order methods. The computational effort is almost the same as required for the lower order methods.

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